

CANONICAL CURVES AND VARIETIES OF SUMS OF POWERS OF CUBIC POLYNOMIALS

ATANAS ILIEV AND KRISTIAN RANESTAD

ABSTRACT. In this note we show that the apolar cubic forms associated to codimension 2 linear sections of canonical curves of genus $g \geq 11$ are special with respect to their presentation as sums of cubes.

1. INTRODUCTION

In a graded Artinian Gorenstein ring A with socle degree d , multiplication defines (up to scalar) a homogeneous form f of degree d , called the socle degree generator, dual polynomial or apolar polynomial of A . Codimension 2 linear sections of a canonical curve of genus g define Artinian Gorenstein quotients of the homogeneous coordinate ring of the curve. These quotients have socledegree 3 and therefore define (up to scalar) cubic forms in $g-2$ variables. A dimension count shows that a general cubic form is not obtained this way when $g \geq 8$. While a general cubic form in $g-2$ variables cannot be written as a sum of less than $\frac{1}{6}g(g-1)$ cubes, our main result says that the cubic forms apolar to a general codimension 2 linear section of a general canonical curve of genus $g \geq 11$ can be written as a sum of $2g-4$ cubes.

Our methods give results concerning the variety of different powersum presentations. In particular we obtain partial results for genus $g = 9$ (cf. 3). Results for $g \leq 6$ are classical, while $g = 7$ and $g = 8$ was treated in [7] and [6].

Powersum presentations of forms from a more algebraic viewpoint have been studied extensively in [5].

We work throughout over the complex numbers \mathbf{C} .

1.1. Powersum presentations. Let $f \in \mathbf{C}[x_0, \dots, x_n]$ be a homogeneous form of degree d , then f can be written as a sum of powers of linear forms

$$f = l_1^d + \dots + l_s^d$$

for s sufficiently large. Indeed, if we identify the map $l \mapsto l^d$ with the d^{th} Veronese embedding $\mathbf{P}^n \hookrightarrow \mathbf{P}^{N_d}$, where $N_d = \binom{n+d}{n} - 1$, this amounts to say that the image spans \mathbf{P}^{N_d} . Fixing (d, n) , the minimal number s of summands needed varies with f , of course. A simple dimension count shows that

$$s \geq \left\lceil \frac{1}{n+1} \binom{n+d}{n} \right\rceil$$

for a **general** f . With a few exceptions equality holds by a result of Alexander and Hirschowitz [1] combined with Terracini's Lemma (cf. [4]) :

⁰Mathematics Subject Classification 13H10 (Primary), 13E10, 14M05, 14H45 (Secondary)

THEOREM 1.1. (Alexander, Hirschowitz) *A general form f of degree d in $n + 1$ variables is a sum of $\lceil \frac{1}{n+1} \binom{n+d}{n} \rceil$ powers of linear forms, unless*

$d = 2$, where $s = n + 1$ instead of $\lceil \frac{n+2}{2} \rceil$, or

$d = 4$ and $n = 2, 3, 4$, where $s = 6, 10, 15$ instead of $5, 9, 14$ respectively, or

$d = 3$ and $n = 4$, where $s = 8$ instead of 7 .

Let $F = Z(f) \subset \mathbf{P}^n$ be the hypersurface defined by f . For a linear form l we denote by L the point in $\check{\mathbf{P}}^n$ of the hyperplane $Z(l) \subset \mathbf{P}^n$. Then we define, as in [7], the variety of sums of powers as the closure

$$VSP(F, s) = \overline{\{ \{L_1, \dots, L_s\} \in \text{Hilb}_s(\check{\mathbf{P}}^n) \mid \exists \lambda_i \in \mathbf{C} : f = \lambda_1 l_1^d + \dots + \lambda_s l_s^d \}}$$

of the set of powersums presenting f in the Hilbert scheme (cf. [7]). Notice that taking d^{th} roots of the λ_i , we can put them into the forms l_i . We study these varieties of sums of powers using apolarity.

1.2. Apolarity. (cf. [7]). Consider $R = \mathbf{C}[x_0, \dots, x_n]$ and $T = \mathbf{C}[\partial_0, \dots, \partial_n]$. T acts on R by differentiation:

$$\partial^\alpha \cdot x^\beta = \alpha! \binom{\beta}{\alpha} x^{\beta-\alpha}$$

if $\beta \geq \alpha$ and 0 otherwise. Here α and β are multi-indices, $\binom{\beta}{\alpha} = \prod \binom{\beta_i}{\alpha_i}$ and so on. One can interchange the role of R and T by defining

$$x^\beta \cdot \partial^\alpha = \beta! \binom{\beta}{\alpha} \partial^{\alpha-\beta}.$$

This action defines a perfect pairing between forms of degree d and homogeneous differential operators of order d . In particular, R_1 and T_1 are natural dual vector spaces. Therefore the projective spaces with coordinate ring R and T respectively are natural dual to each other, we denote them by \mathbf{P}^n and $\check{\mathbf{P}}^n$. A point $a = (a_0, \dots, a_n) \in \check{\mathbf{P}}^n$ defines a form $l_a = \sum a_i x_i \in R_1$, and for a form $D \in T_e$

$$D \cdot l_a^d = e! \binom{d}{e} D(a) l_a^{d-e},$$

when $e \leq d$. In particular

$$(*) \quad D \cdot l_a^d = 0 \iff D(a) = 0$$

if $e \leq d$. More generally we say that homogeneous forms $f \in R$ and $D \in T$ are **apolar** if $f \cdot D = D \cdot f = 0$ (According to Salmon (1885) [8] the term was coined by Reye).

Apolarity allows us to associate an Artinian Gorenstein graded quotient ring of T to a form: For $f \in R$ a homogeneous form of degree d and $F = Z(f) \subset \mathbf{P}^n$ define

$$F^\perp = f^\perp = \{D \in T \mid D \cdot f = 0\}$$

and

$$A^F = T/F^\perp.$$

The socledegree of A^F is d , since

$$D' \cdot (D \cdot f) = 0 \quad \forall D' \in T_1 \iff D \cdot f = 0 \text{ or } D \in T_d.$$

In particular the socle of A^F is 1-dimensional, and A^F is Gorenstein. It is called the apolar Artinian Gorenstein ring of F . Conversely for a graded Gorenstein ring $A = T/I$ with socledegree d , multiplication in A induces a linear form $f: \text{Sym}_d(T_1) \rightarrow \mathbf{C}$

which can be identified with a homogeneous polynomial $f \in R$ of degree d . This proves:

LEMMA 1.2. (Macaulay, [2]) *The map $F \mapsto A^F$ is a bijection between hypersurfaces $F = Z(f) \subset \mathbf{P}^n$ of degree d and graded Artinian Gorenstein quotient rings $A = T/I$ of T with socledegree d .*

Let $X \subset \mathbf{P}^{n+m+1}$ be a m -dimensional arithmetic Gorenstein variety. Let $S(X)$ be the homogeneous coordinate ring of X , and let h_1, \dots, h_{m+1} be general linear forms and set $L = Z(h_1, \dots, h_{m+1})$. Then by definition $S(X)/(h_1, \dots, h_{m+1})$ is Artinian Gorenstein, i.e. by Macaulay's result the apolar Artinian Gorenstein ring of a $(n-1)$ -dimensional hypersurface F_L of degree d , the socledegree of the ring. L is a linear space of dimension n and by apolarity $F_L = Z(f_L)$ is a hypersurface in the dual space to L . We say that F_L is apolar to the (empty) linear section $L \cap X$. Hence, there is a rational map

$$\alpha_X : \mathbf{G}(n+1, m+n+2) \dashrightarrow H_{n,d}$$

Where $H_{n,d}$ is the space of $(n-1)$ -dimensional hypersurfaces of degree d modulo the action of $PGL(n+1, k)$.

A canonical curve $C \subset \mathbf{P}^{g(C)-1}$ is arithmetic Gorenstein, i.e. the homogeneous coordinate ring $S(C)$ is Gorenstein. Let $h_1, h_2 \in S(C)$ be two general linear forms, then the quotient $S(C)/(h_1, h_2)$ is Artinian Gorenstein with values of the Hilbert function: $1, g-2, g-2, 1$. Its socledegree is therefore 3. Thus we obtain a map

$$\alpha_C : \mathbf{G}(g(C)-2, g(C)) \dashrightarrow H_{g(C)-3,3}$$

to the space of cubic hypersurfaces of dimension $g(C)-4$. We shall study the image of this map. In particular we shall study the variety of sums of powers of the cubic hypersurfaces in this image.

1.3. Variety of apolar subschemes. Let $F = Z(f) \subset \mathbf{P}^n$ denote a hypersurface of degree d . We call a subscheme $\Gamma \subset \check{\mathbf{P}}^n$ **apolar** to F , if the homogeneous ideal $I_\Gamma \subset F^\perp \subset T$.

APOLARITY LEMMA 1.3. *Let l_1, \dots, l_s be linear forms in R , and let $L_i \in \check{\mathbf{P}}^n$ be the corresponding points in the dual space. Then $f = \lambda_1 l_1^d + \dots + \lambda_s l_s^d$ for some $\lambda_i \in \mathbf{C}^*$ if and only if $\Gamma = \{L_1, \dots, L_s\} \subset \check{\mathbf{P}}^n$ is apolar to $F = Z(f)$.*

Proof. Assume $f = \lambda_1 l_1^d + \dots + \lambda_s l_s^d$. If $g \in I_\Gamma$, then $g \cdot l_i^d = 0$ for all i by (*), so by linearity $g \in F^\perp$. Therefore Γ is apolar to F .

For the converse, assume that $I_\Gamma \subset F^\perp$. Then we have surjective maps between the corresponding homogeneous coordinate rings

$$T \rightarrow A_\Gamma = T/I_\Gamma \rightarrow A^F.$$

Consider the dual inclusions of the degree d part of these rings:

$$\text{Hom}(A_d^F, \mathbf{C}) \rightarrow \text{Hom}((A_\Gamma)_d, \mathbf{C}) \rightarrow \text{Hom}(T_d, \mathbf{C}).$$

$D \mapsto D \cdot f$ generates the first of these spaces, while the second is spanned by the forms $D \mapsto D \cdot l_i^d$. Thus f lies in the span of the l_i^d . \square

This is the crucial lemma in the study of powersum presentations of f . Furthermore it allows us to define a **variety of apolar subschemes** to f , which naturally extends our definition of the variety of sums of powers.

$$VPS(F, s) = \overline{\{\Gamma \in \text{Hilb}_s(\check{\mathbf{P}}^n) \mid I_\Gamma \subset F^\perp\}},$$

where $\text{Hilb}_s(\check{\mathbf{P}}^n)$ is the Hilbert scheme of length s subschemes of $\check{\mathbf{P}}^n$. Clearly $VSP(F, s)$ is the closure of the set parametrizing smooth subschemes in $VPS(F, s)$. In general they do not coincide.

2. APOLAR VARIETIES OF SINGULAR SECTIONS

2.1. Apolar varieties. Let $X \subset \mathbf{P}^{n+m+1}$ be a reduced and irreducible m -dimensional nondegenerate variety of degree $d \geq 3$ and codimension $n+1 \geq 2$. Let $p \in X$ be a general smooth point. Let $C_p X$ be the cone over X with vertex at p . Since p is a smooth point, the degree of the cone $C_p X$ is $d-1$, while the dimension is $m+1$. Clearly $X \subset C_p X$.

We apply this simple construction to describe powersum presentations of hypersurfaces in the image of the map α_X in 1.2. Let again $X \subset \mathbf{P}^{n+m+1}$ be a m -dimensional arithmetic Gorenstein variety of degree d . Fix a general n -dimensional linear subspace $L \subset \mathbf{P}^{n+m+1}$, in particular we fix the hypersurface F_L in the image of α_X . Let p be a smooth point on X , then the intersection $C_p X \cap L$ is clearly nonempty, and if it is proper it is 0-dimensional of degree $d-1$. We may assume that this intersection is proper and smooth for a general L or general p , so we get an apolar subscheme of degree $d-1$ to F_L , i.e. a point in $VSP(F_L, d-1)$. We have shown:

PROPOSITION 2.1. *Let $X \subset \mathbf{P}^{n+m+1}$ be a m -dimensional arithmetic Gorenstein variety of degree d , and let $L \subset \mathbf{P}^{n+m+1}$ be a n -dimensional linear subspace such that $L \cap X = \emptyset$. Let F_L be the associated apolar hypersurface. Then there is a rational map $X \dashrightarrow VSP(F_L, d-1)$ defined by $p \mapsto C_p X \cap L$.*

Problem 2.2. *When is this map a morphism? When can F_L and X be recovered from the image of this map?*

We may improve slightly on the degree of the apolar subschemes by considering cones on special linear sections of X .

2.2. Tangent hyperplane sections. Let $X \subset \mathbf{P}^{n+m+1}$ be a reduced and irreducible m -dimensional nondegenerate variety of degree d and codimension $n+1 \geq 2$. We assume additionally that X satisfies the following condition:

- (**) A general tangent hyperplane section of X has a double point at the point of tangency, and the projection of the tangent hyperplane section from the point of tangency is birational.

In particular, X is not a scroll and $d \geq 4$. Let $p \in X$ be a general smooth point. Let H_p be a general hyperplane tangent to X at p . Since $H_p \cap X$ has multiplicity 2 at p and the projection of $H_p \cap X$ from p is birational, the image of the projection is $(m-1)$ -dimensional of degree $d-2$. Therefore $H_p \cap X$ is contained in an m -dimensional cone $C_p(H_p \cap X)$ of degree $d-2$ with vertex at p . Similarly, if H_p and H'_p are two general hyperplanes tangent at p , then the intersection $H_p \cap H'_p \cap X$ has a singularity at p of multiplicity 4, the complete intersection of two singularities of multiplicity 2. In this case we say that the codimension 2 space $H_p \cap H'_p$ is **doubly tangent** to X at p . If

- (***) the projection of $H_p \cap H'_p \cap X$ from p is birational,

then the image $(m-2)$ -dimensional of degree 4 less than the degree of X . Hence $H_p \cap H'_p \cap X$ is contained in a $(m-1)$ -dimensional cone $C_p(H_p \cap H'_p \cap X)$ of degree $d-4$ with vertex at p . This proves the

LEMMA 2.3. *Let $X \subset \mathbf{P}^{n+m+1}$ be a smooth m -dimensional nondegenerate variety of degree d , and assume that X satisfies condition (**). Let $p \in X$ be a general smooth point, and let H_p be a general hyperplane tangent to X at p . Then the cone $C_p(H_p \cap X)$ is an m -dimensional variety of degree $d-2$ that contains $H_p \cap X$. Assume furthermore that X satisfies condition (***), and let H_p and H'_p be two general hyperplanes tangent to X at p . Then the cone $C_p(H_p \cap H'_p \cap X)$ is a $(m-1)$ -dimensional variety of degree $d-4$ that contains $H_p \cap H'_p \cap X$.*

As above we apply this lemma to describe powersum presentations of hypersurfaces in the image of the map α_X in section 1.2. Let again $X \subset \mathbf{P}^{n+m+1}$ be a m -dimensional arithmetic Gorenstein variety of degree d , with $m \geq 1$. We assume additionally that X satisfies condition (**). Fix a general n -dimensional linear subspace $L \subset \mathbf{P}^{n+m+1}$, in particular we fix the hypersurface F_L in the image of α_X . If $L \subset H_p$ where H_p is a general hyperplane tangent at p , then according to 2.1 there is a $(m-1)$ -dimensional variety $Y \supset H_p \cap X$ of degree $d-2$. The intersection $Y \cap L$ is clearly nonempty, and if it is proper it is 0-dimensional of degree $d-2$. We may assume that this intersection is proper for a general L , so we get a point in $VSP(F_L, d-2)$. Let $\check{X} \subset \check{\mathbf{P}}^{m+n+1}$ be the dual variety of X , i.e. the set of hyperplanes tangent at some point $p \in X$. Then we have set up a rational map

$$\check{X}_L \rightarrow VSP(F_L, d-4)$$

where $\check{X}_L = \{[H] \in \check{X}_L | H \supset L\}$. The subvariety \check{X}_L has dimension $m - \text{codim} \check{X}$, which equals $m-1$ when \check{X} is nondegenerate. In particular this is the case when X is a curve.

Similarly, assume that X satisfies (***), and let $L \subset H_p \cap H'_p$ where H_p and H'_p are two general hyperplanes tangent at p . Then, according to 2.1, there is a $(m-1)$ -dimensional variety

$$Y \supset H_p \cap H'_p \cap X$$

of degree $d-4$. The intersection $Y \cap L$ is clearly nonempty, and if it is proper it is 0-dimensional of degree $d-4$. We may assume that this intersection is proper for a general L , so we get a point in $VSP(F_L, d-4)$. Let $Z_X \subset \mathbf{G}(m+n, m+n+2)$ be the set of codimension 2 subspaces doubly tangent at some point $p \in X$. Then we have set up a rational map

$$\mathbf{G}(m+n, m+n+2) \supset Z_L \dashrightarrow VSP(F_L, d-4)$$

where $Z_L = \{[V] \in Z_X | V \supset L\}$. If the dual variety of X is nondegenerate, then the subvariety Z_X has dimension $m+2(n-1)$. The codimension in $\mathbf{G}(m+n, m+n+2)$ of subspaces that contains L is $2(m+n) - 2(m-1) = 2n+2$ so the expected dimension of Z_L is $m-4$.

Notice that it is essential for the dimension count that X is not a cone, i.e. that the dual variety is nondegenerate.

PROPOSITION 2.4. *Let $X \subset \mathbf{P}^{n+m+1}$ be a m -dimensional arithmetic Gorenstein variety of degree d , with $m \geq 1$. Assume that X satisfies the condition (**) and has nondegenerate dual variety. Let $L \subset \mathbf{P}^{n+m+1}$ be a general n -dimensional linear subspace, and let $F_L = \alpha_X([L])$ be the hypersurface apolar to $L \cap X$. Then*

$VSP(F_L, d-2) \neq \emptyset$ and of dimension at least $m-1$. Assume furthermore that $m \geq 4$ and that X also satisfies condition $(***)$. Then the dimension of $VSP(F_L, d-4)$ is at least $m-4$, when $m \geq 4$ and L is contained in at least one codimension 2 linear space doubly tangent to X .

3. CANONICAL CURVES AND APOLAR CUBIC POLYNOMIALS

For a general cubic n -fold F , the result of Alexander and Hirschowitz (1.1) implies that $VSP(F, k) = \emptyset$, when $k < \frac{1}{6}(n+4)(n+3)$. In 1.2 we defined a map α_C that associates an apolar cubic n -fold to an empty codimension two linear section of a canonical curve C of genus $g = n+4$. The following theorem shows that cubic n -folds in the image of this map are special with respect to the possible powersum presentations as soon as $n \geq 7$.

THEOREM 3.1. *If F is a cubic n -fold apolar to a general codimension two linear section of a general canonical curve of genus $g = n+4$, then $VSP(F, 2n+4) \neq \emptyset$.*

Proof. This is immediate from 2.4 since a canonical curve has nondegenerate dual variety and satisfies $(**)$. \square

REMARK 3.2. *By Hurwitz' formula, the degree of the dual variety of a canonical curve is $6g-6$, so $VSP(F, 2n+4)$ contains at least $6n+18$ points. We do not know whether there are more.*

For $n \leq 3$, the general cubic is apolar to a section of a canonical curve. This fact can be used to describe completely the powersum presentations of the cubic form (cf. [7]).

For $n = 3, 4, 5$ the general canonical curve of genus $g = n+4$ is a linear section of a homogeneous space of dimension at least 6 (cf. [3]). For $n = 4, 5$, these homogeneous spaces of dimension 8 (resp. 6) have nondegenerate dual varieties. For a cubic 4-fold F apolar to a general canonical curve of genus 8 proposition 2.4 gives us a 4-dimensional component of $VSP(F, 10)$. It is shown in [6] that this is in fact all of $VSP(F, 10)$.

A general canonical curve of genus 9 is a linear section of the symplectic grassmannian $Sp(3)/U(3) \subset \mathbf{G}(3, 6)$. For a cubic 5-fold F apolar to a canonical curve of genus 9, which is contained in a codimension two linear section doubly tangent to $Sp(3)/U(3)$, proposition 2.4 gives us a 2-dimensional subvariety of $VSP(F, 12)$. On the other hand, for a general cubic 5-fold F it follows from 1.1 that $VSP(F, 12)$ is finite.

The general canonical curve of genus 10 is not a section of a $K3$ -surface (cf. [3]), so only 3.1 applies i.e. $VSP(F, 16) \neq \emptyset$, while already $VSP(F, 15) \neq \emptyset$ for a general cubic 6-fold F .

REFERENCES

- [1] Alexander, J., Hirschowitz, A.: Polynomial interpolation in several variables, J. of Alg. Geom. **4** (1995), 201-222
- [2] Macaulay, F.S.: Algebraic theory of modular systems. Cambridge University Press, London, (1916)
- [3] Mukai, S.: Curves, $K3$ surfaces and Fano 3-folds of genus ≤ 10 . "Algebraic Geometry and Commutative Algebra in Honor of Masayoshi Nagata", pp. 357-377, (1988), Kinokuniya, Tokyo
- [4] Iarrobino, A.: Inverse systems of a symbolic power. II: The Waring problem for forms. J. of Algebra, **174** (1995), 1091-1110

- [5] Iarrobino A., Kanev V.: Power sums, Gorenstein algebras, and determinantal loci. Lecture notes in mathematics **1721**, Springer Verlag, Berlin (1999)
- [6] Iliev, A., Ranestad, K.: $\mathbf{K3}$ surfaces of genus 8 and varieties of sums of powers of cubic fourfolds. Trans. Amer. Math. Soc. **353** (2001), 1455-1468.
- [7] Ranestad, K., Schreyer, F.-O: Varieties of sums of powers. J. reine angew. Math. **525** (2000), 147-181
- [8] Salmon, G.: Modern Higher Algebra, 4. Edition. Hodges, Figgis, and Co., Dublin (1885)

Authors' addresses:

Atanas Iliev
Institute of Mathematics, Bulgarian Academy of Sciences,
Acad. G. Bonchev Str., 8,
1113 Sofia, Bulgaria.
e-mail: ailiev@math.bas.bg

Kristian Ranestad
Matematisk Institutt, UiO,
P.B. 1053 Blindern,
N-0316 Oslo, Norway.
e-mail: ranestad@math.uio.no